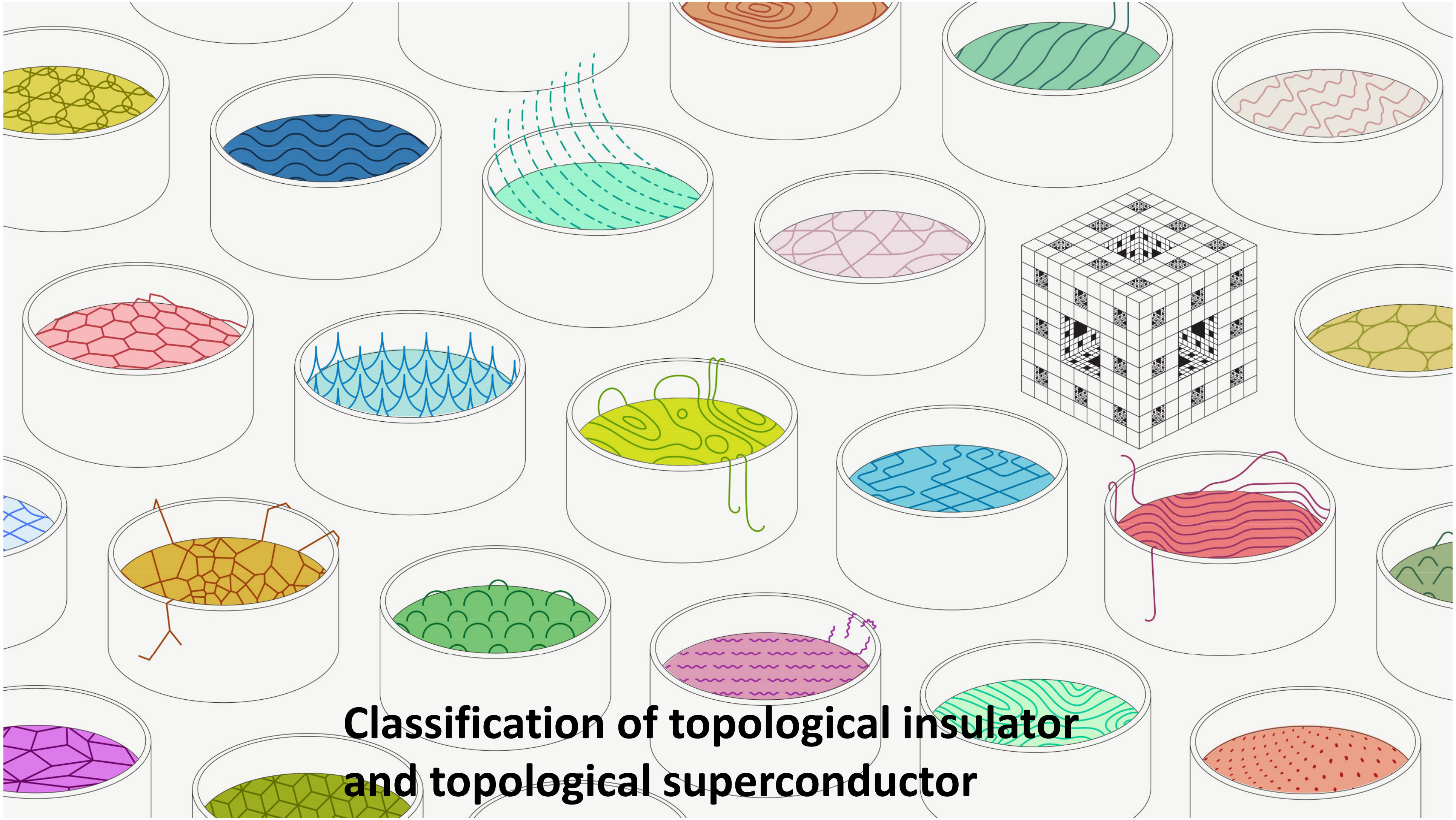
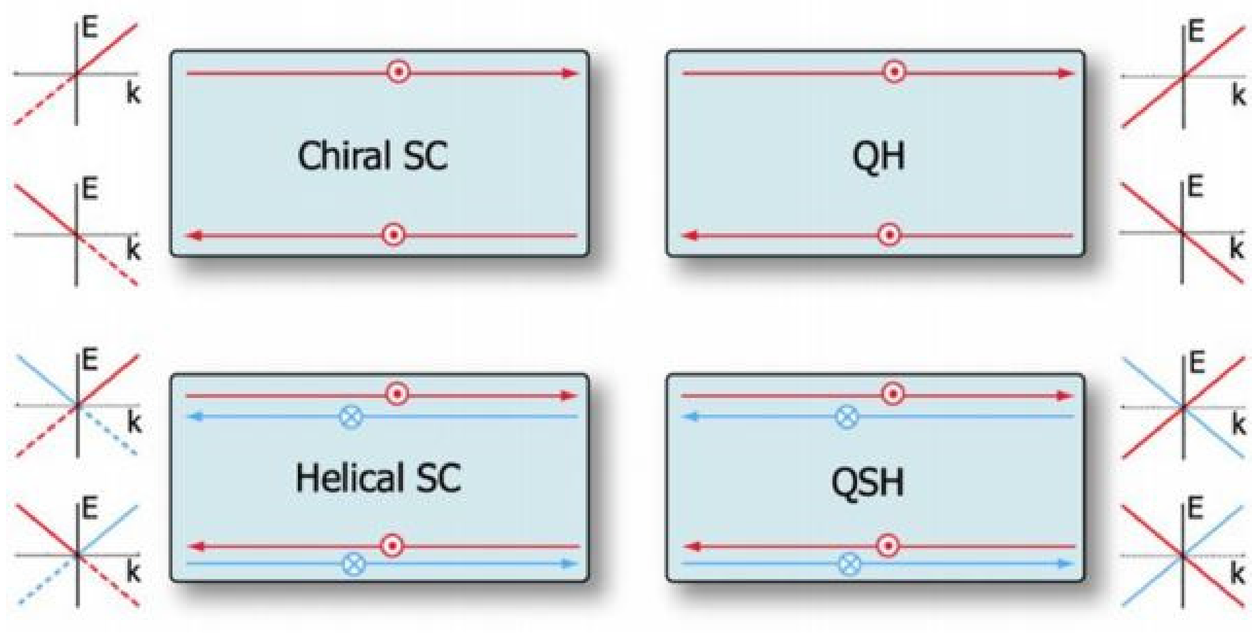


Classification of topological insulator and topological superconductor



	1d	2d	3d	T	P	S	Lect
Quantum Hall insulator	0	Z	0	0	0	0	4
Topological insulator	0	Z_2	Z_2	-1	0	0	6,7
Chiral superconductor	Z_2	Z	0	0	1	0	16,17
Helical superconductor	Z_2	Z_2	Z	-1	1	1	18,19

Periodic table: different approaches

1. Continuous systems: Dirac Hamiltonian, Clifford algebra

   Bernard and LeClair, J Phys A 2002

2. Disordered systems: Surface state localization, random matrix theory, nonlinear sigma model Ivanov, 9911147, Schnyder et al PRB 2008, Ryu et al, NJP 2010

3. Lattice systems: Homotopy theory Schnyder et al PRB 2008,

   K-theory Kitaev AIP Conf Proc 2009

   Related to classification of symmetric spaces (Cartan 1926-27)

4. Response theory, quantum anomaly Ryu et al, PRB 2012

5. …

PHYSICAL REVIEW VOLUME 109, NUMBER 5 MARCH 1, 1958

Absence of Diffusion in Certain Random Lattices

P. W. ANDERSON
Bell Telephone Laboratories, Murray Hill, New Jersey

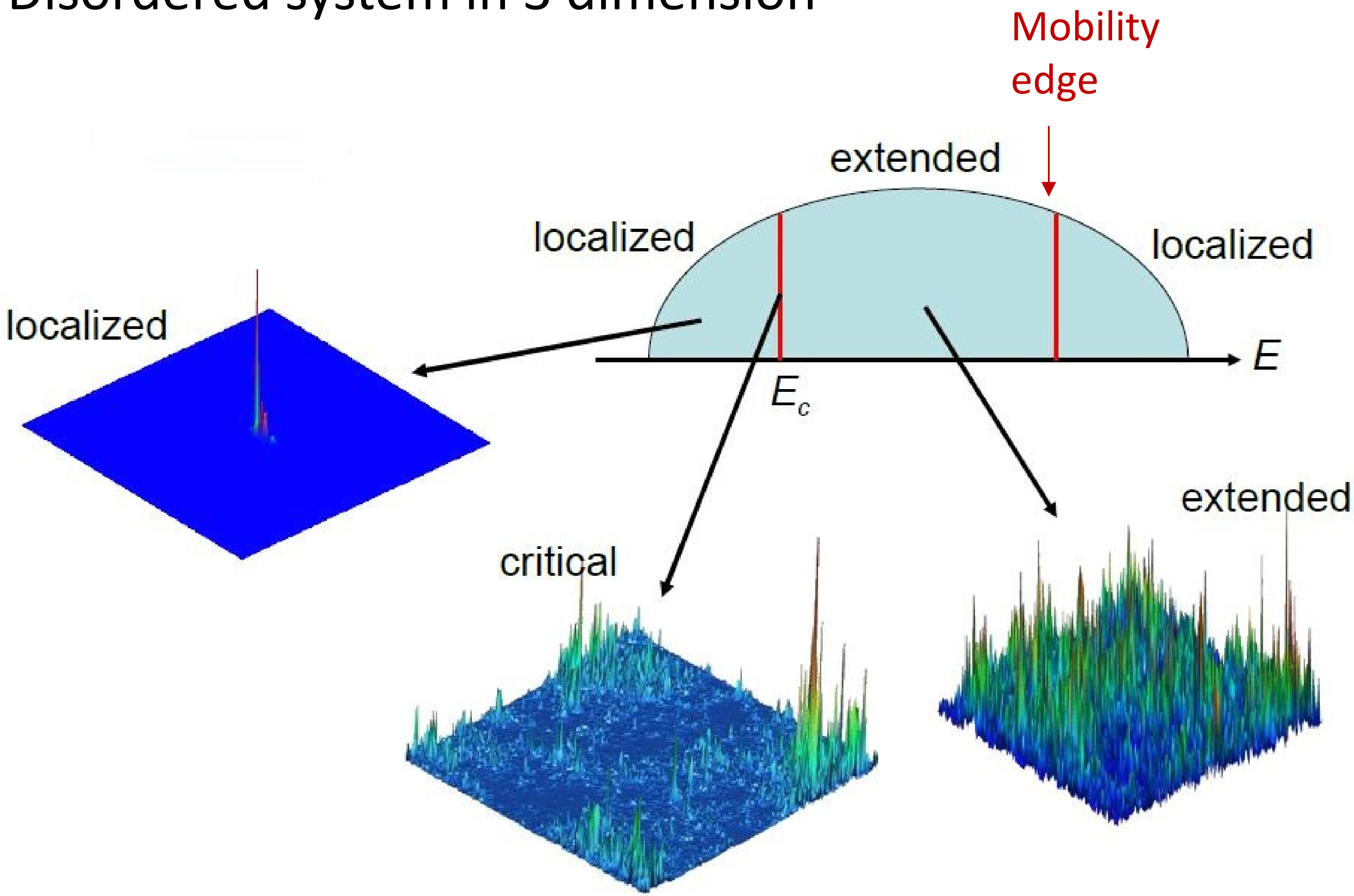
(Received October 10, 1957)

This paper presents a simple model for such processes as spin diffusion or conduction in the "impurity band." These processes involve transport in a lattice which is in some sense random, and in them diffusion is expected to take place via quantum jumps between localized sites. In this simple model the essential randomness is introduced by requiring the energy to vary randomly from site to site. It is shown that at low enough densities no diffusion at all can take place, and the criteria for transport to occur are given.

In the modern literature, the phenomenon of exponential decay of eigenfunctions of a quantum system in a disordered environment is called Anderson localization,

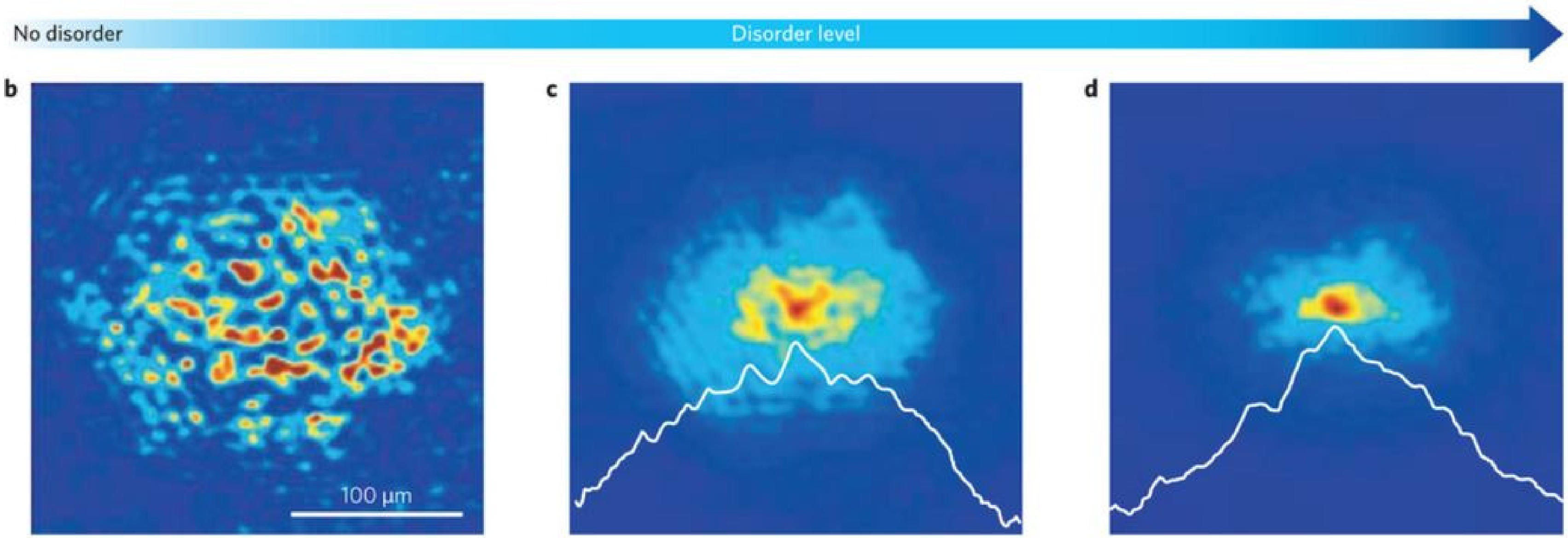
Chulaevsky and Suhov. Multi-scale Analysis for Random Quantum Systems with Interaction

Disordered system in 3 dimension



Anderson localization of light:

Transition from ballistic transport to diffusive transport, to Anderson localization.



Nature Photonics 7, 197 (2013)

- Weak localization
(Gorkov et al, 1979)

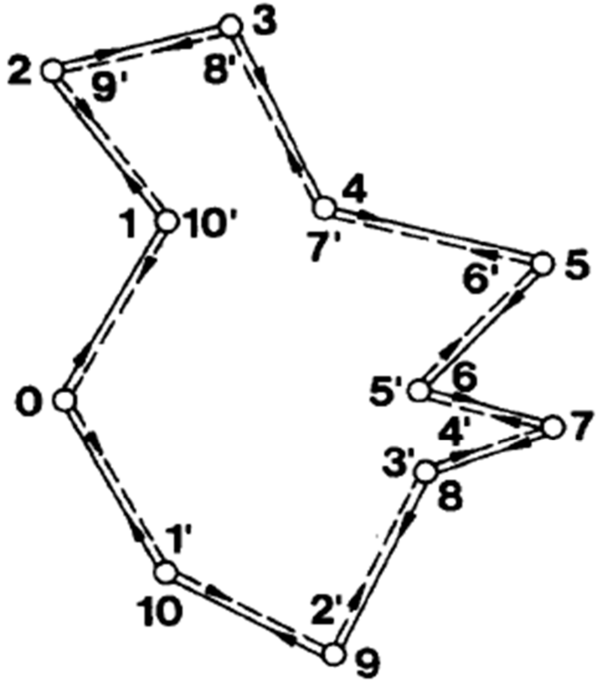


Fig. 2.5. Diffusion path of the conduction electron in the disordered system. The electron propagates in both directions (full and dashed lines). In the case of quantum diffusion the probability to return to the origin is twice as great as in classical diffusion since the amplitudes add coherently.

Bergmann Phys Rep 1984

Transition amplitude

$$A_{a\to b} = \sum_i A_{path\ i}$$

$$|A_{a\to b}|^2 = \sum_i |A_i|^2 + \sum_{i\neq j} A_i A_j^*$$

$$\left\langle \sum_{i\neq j} A_i A_j^* \right\rangle \neq 0 \quad \text{for time-reversed path } A_i = \bar{A}_j$$

$$(\text{possible only when } a = b)$$

$$\to \quad |A_{a\to a}|^2 = 2\sum_i |A_i|^2$$

coherent back-scattering

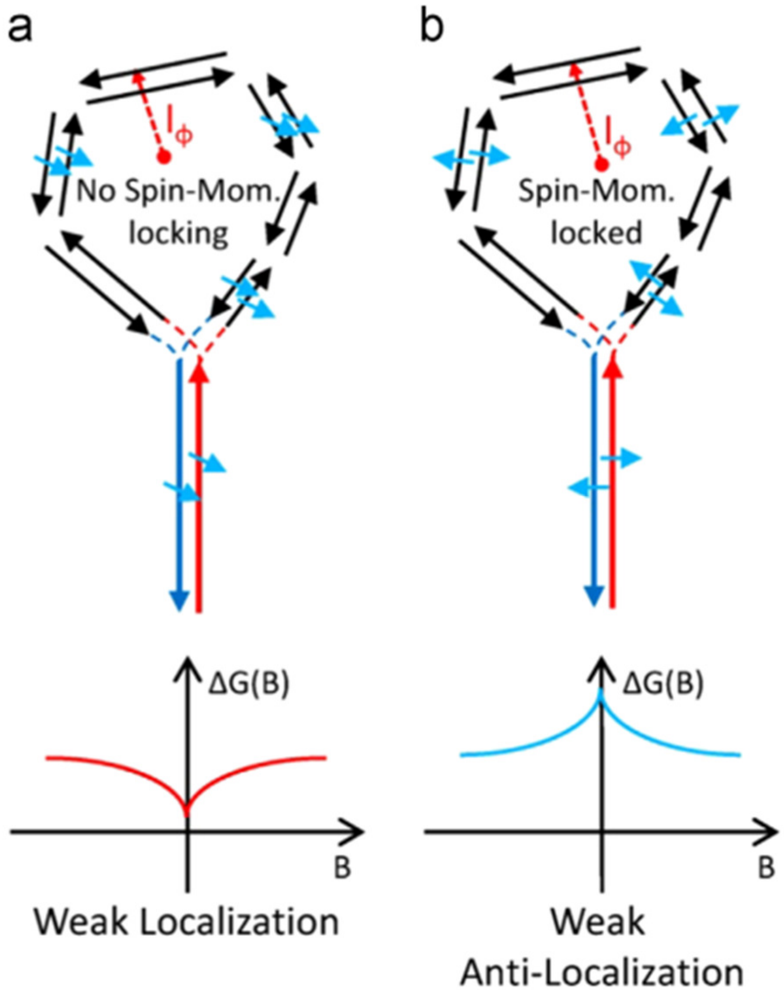
(Constructive interference)

Note: elastic scattering (static disorder) does not destroy quantum coherence (inelastic: phonons, other electrons… etc)

Weak localization vs weak anti-localization

(Hikami et al, 1980)

Effect of magnetic field



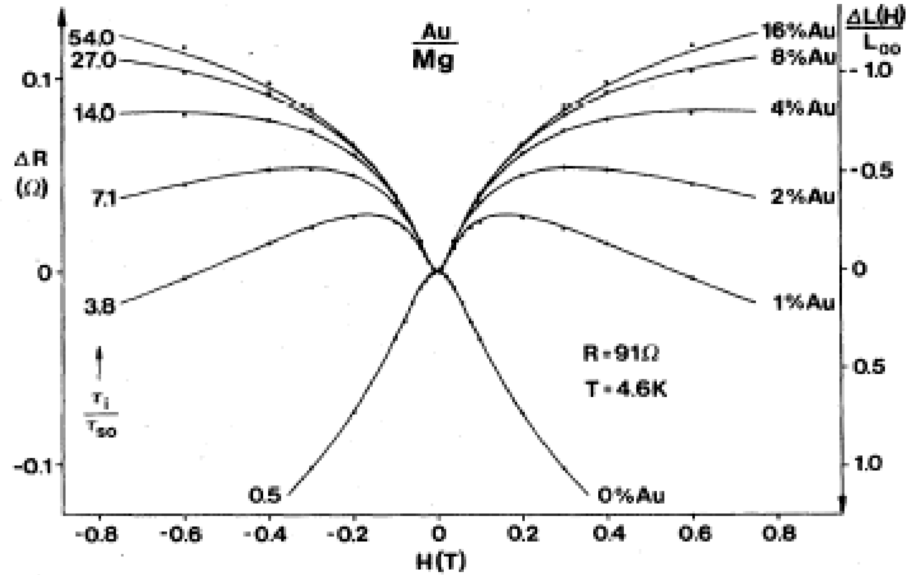


FIG. 17. The magneto~~conductance~~ curve of a Mg film with different coverages of Au. [$\Delta L(H)$ is the magnetoconductance, and $L_{\infty}=e^2/2\pi^2\hbar$.] The coverages shown are in percent of an atomic layer. Increasing Au coverage converts the positive magnetoconductance to negative. Full curves through the data points are fits using the theory of Hikami, Larkin, and Nagaoka (1980). Figure is taken from Bergmann (1982b).

Fig: Brahlek et al, Solid State Comm 2015

VOLUME 42, NUMBER 10 PHYSICAL REVIEW LETTERS 5 MARCH 1979

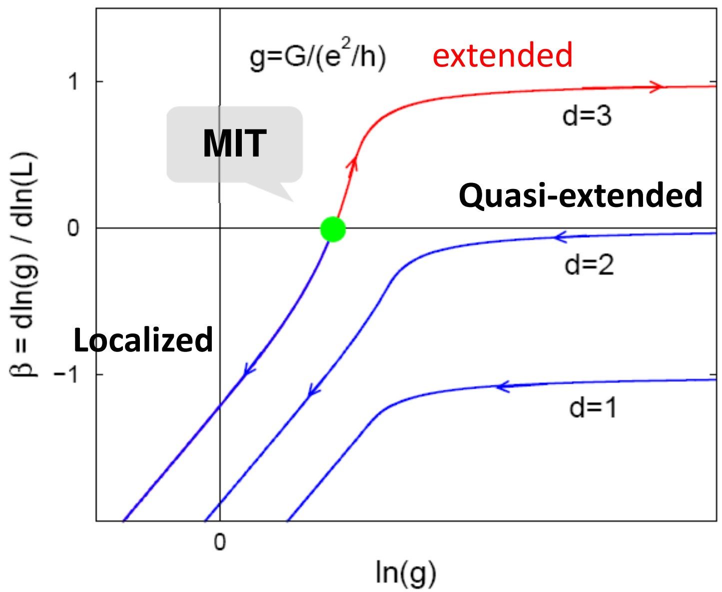
Scaling Theory of Localization: Absence of Quantum Diffusion in Two Dimensions

E. Abrahams
Serin Physics Laboratory, Rutgers University, Piscataway, New Jersey 08854

and

P. W. Anderson,[a] D. C. Licciardello, and T. V. Ramakrishnan[b]
Joseph Henry Laboratories of Physics, Princeton University, Princeton, New Jersey 08540
(Received 7 December 1978)

- one-parameter scaling hypothesis



Flow follows the increase of L

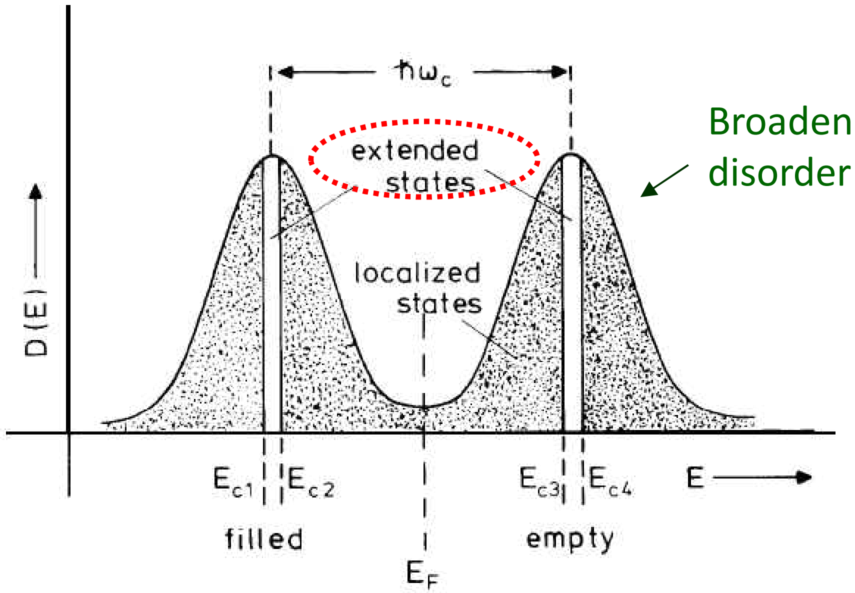
→ • All wave functions of disordered systems in 1D and 2D are localized
- Exception (in 2D): Quantum Hall effect, spin-orbit interaction

Lagendijk et al, Phys Today 2009

[Exception 1]

Quantum Hall effect (1980)

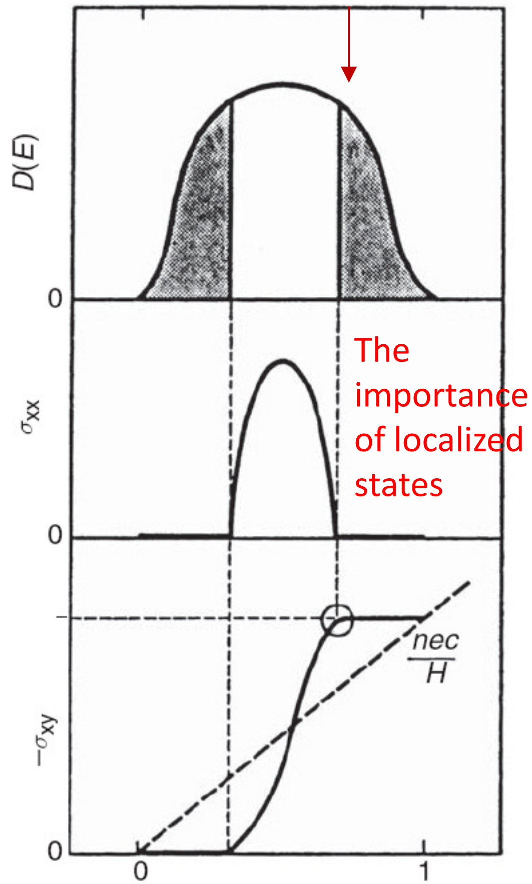
Localized states vs extended states



Broadened LL due to disorder

Aoki, CMST 2011

Mobility edge



Filling factor

Abraham *et al*'s conclusion does not apply
(since QHE is in a different universality class)

- Extended state -><- earlier conclusion. Why?

Prog. Theor. Phys. Vol. 63, No. 2, February 1980, Progress Letters

[Exception 2]

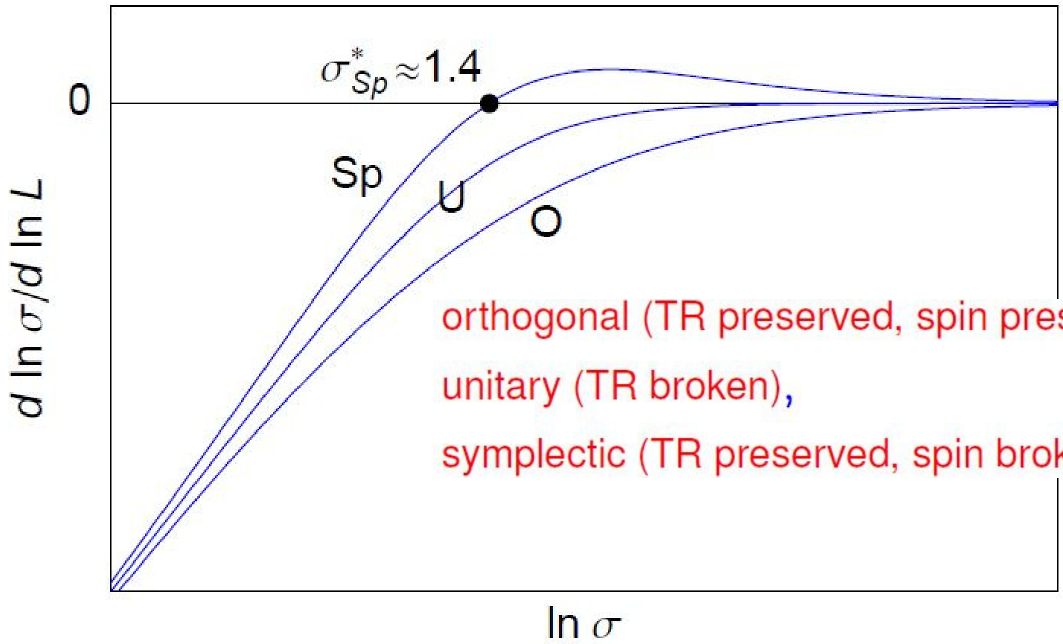
Spin-Orbit Interaction and Magnetoresistance in the Two Dimensional Random System

氷上忍 Shinobu HIKAMI, Anatoly I. LARKIN*) and Yosuke NAGAOKA 長岡洋介

Research Institute for Fundamental Physics
Kyoto University, Kyoto 606

(Received November 5, 1979)

Effect of the spin-orbit interaction is studied for the random potential scattering in two dimensions by the renormalization group method. It is shown that the localization behaviors are classified in the three different types depending on the symmetry. The recent observation of the negative magnetoresistance of MOSFET is discussed.



orthogonal (TR preserved, spin preserved),
unitary (TR broken),
symplectic (TR preserved, spin broken)

Ostrovsky's slide

Connection with Random matrix theory

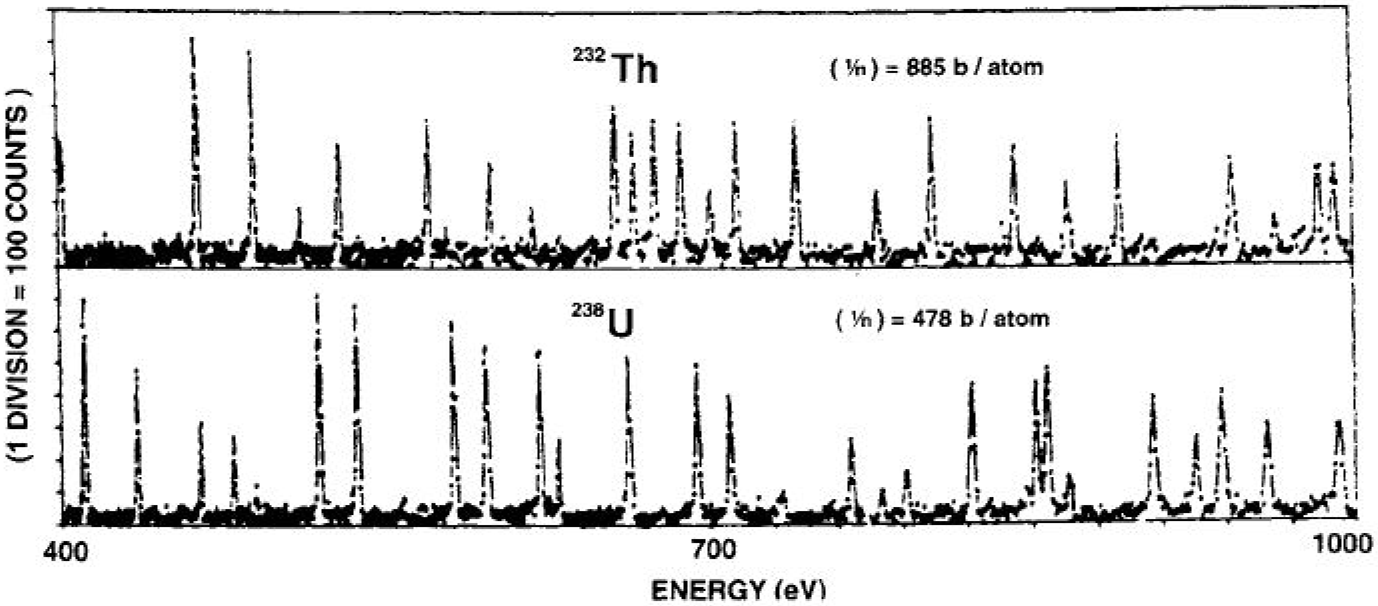
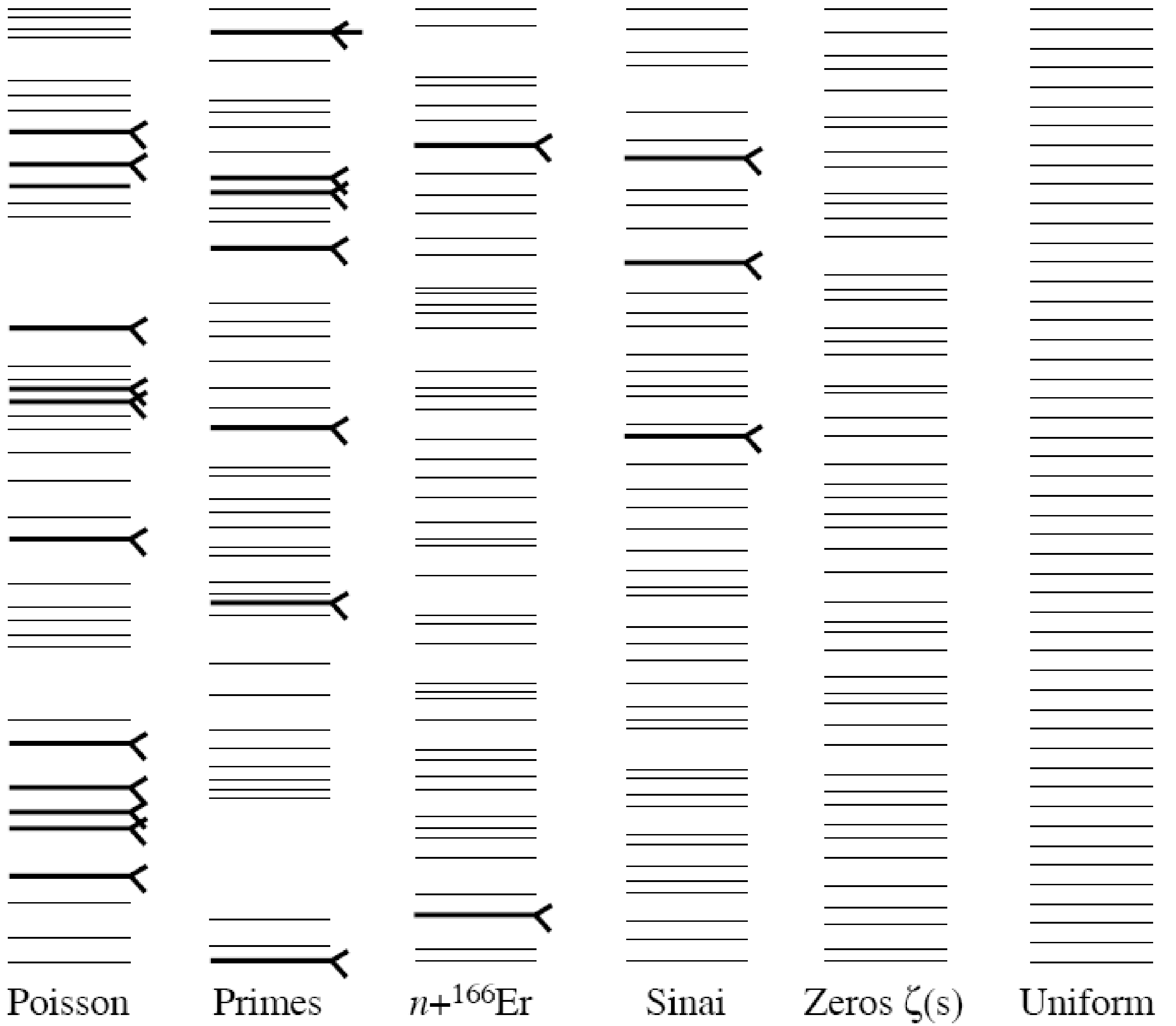


Figure 1.1. Slow neutron resonance cross-sections on thorium 232 and uranium 238 nuclei. Reprinted with permission from The American Physical Society, Rahn et al., Neutron resonance spectroscopy, X, *Phys. Rev. C* 6, 1854–1869 (1972).

Problem : excitation spectrum of heavy nuclei
many-body problem; do not know Hamiltonian

Solution : write Hamiltonian as random matrix

Wigner

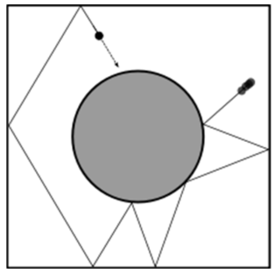


"Random" Gaps. *The statistics of nearest-neighbor spacings range from random to uniform (<'s indicate spacings too close for the figure to resolve). The second column shows the primes from 7,791,097 to 7,791,877. The third column shows energy levels for an excited heavy (Erbium) nucleus. The fourth column is a "length spectrum" of periodic trajectories for Sinai billiards. The fifth column is a spectrum of zeroes of the Riemann zeta function. (Figure courtesy of Springer-Verlag New York, Inc., "Chaotic motion and random matrix theories" by O. Bohigas and M. J. Giannoni in* Mathematical and Computational Methods in Nuclear Physics, *J. M. Gomez et al., eds., Lecture Notes in Physics, volume 209 (1984), pp. 1–99.)*

- Poisson

$$\Pr(X=k)=\frac{\lambda^k e^{-\lambda}}{k!}$$

- Sinai



- Riemann

$$\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{p\text{ prime}}\frac{1}{1-p^{-s}}$$

3.4. The setting. — Dyson now considers a general Hilbert space V with a group G acting on it by unitary and anti-unitary operators. The physical meaning of the group G is that of the group of symmetries of the particular quantum system which is to be treated statistically by a random matrix model. The symmetry group G is meant to be quite arbitrary; in Dyson's words, it "may be a rotation group, or an isotopic-spin rotation group, or a time-inversion group, or all of these in combination."

Needless to say, the 'good' Hamiltonians are those that commute (in the sense just described) with all of the symmetry operations from G. The question (pointedly asked by Dyson) then is: what can be said about the structure of the set of all good Hamiltonians, which are compatible with these symmetry constraints?

Zirnbauer's 2011 talk at Muenchen

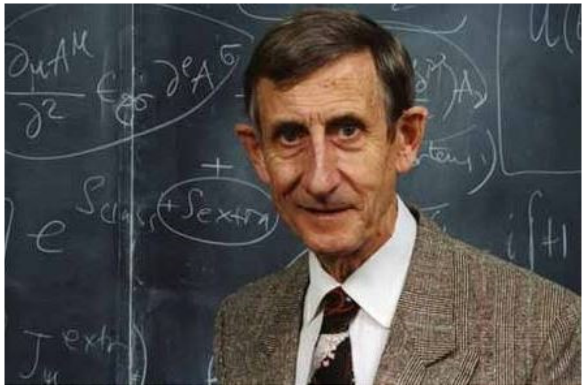
JOURNAL OF MATHEMATICAL PHYSICS VOLUME 3, NUMBER 6 NOVEMBER-DECEMBER 1962

The Threefold Way. Algebraic Structure of Symmetry Groups and Ensembles in Quantum Mechanics

FREEMAN J. DYSON
Institute for Advanced Study, Princeton, New Jersey
(Received June 22, 1962)

Using mathematical tools developed by Hermann Weyl, the Wigner classification of group-representations and co-representations is clarified and extended. The three types of representation, and the three types of co-representation, are shown to be directly related to the three types of division algebra with real coefficients, namely, the real numbers, complex numbers, and quaternions. The author's theory of matrix ensembles, in which again three possible types were found, is shown to be in exact correspondence with the Wigner classification of co-representations. In particular, it is proved that the most general kind of matrix ensemble, defined with a symmetry group which may be completely arbitrary, reduces to a direct product of independent irreducible ensembles each of which belongs to one of the three known types.

Orthogonal, unitary, symplectic



Wigner-Dyson classes

TABLE I. Summary of Dyson's threefold way. The Hermitian matrix $\mathcal{H}$ (and its matrix of eigenvectors U) are classified by an index $\beta \in \{1,2,4\}$, depending on the presence or absence of time-reversal (TRS) and spin-rotation (SRS) symmetry.

β	TRS	SRS	$\mathcal{H}_{nm}$	U	
1	yes	yes	real	orthogonal	**GOE: T^2=1**
2	no	irrelevant	complex	unitary	**GUE: T^2=0**
4	yes	no	real quaternion	symplectic	**GSE: T^2=-1**

Distribution of NN spacing

level repulsion: $P(s \ll 1) \sim s^{\beta}$

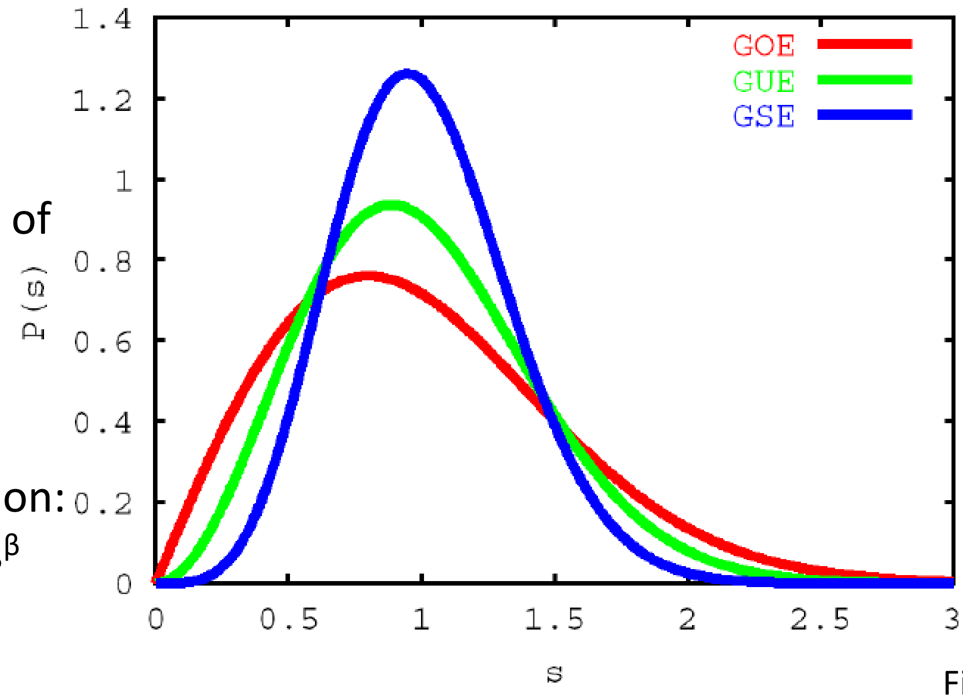


Fig from Altshuler's ppt

In the first half of the 1990's there was quite some activity to work out the density of states for a system with disorder. And although all groups agreed that they were addressing exactly the same physical problem, there was a cacophony of different predictions for the density of states, varying from vanishing linearly to vanishing with a disorder-dependent exponent, to finite (at zero energy), to logarithmically divergent.

4.7. Resolution. — When the controversy did not abate, we decided to write a Physics Report with Alex Altland and Ben Simons, where we pointed out that all proposals but one were inappropriate because they derived from models which belonged to symmetry classes different from the particular symmetry class of the problem at hand, which is CI.

This case study once again underlined a point made by Wigner and Dyson: that it is crucial to understand what is the symmetry class of the problem you are looking at.

Zirnbauer's 2011 talk at Muenchen

Altland-Zirnbauer classes

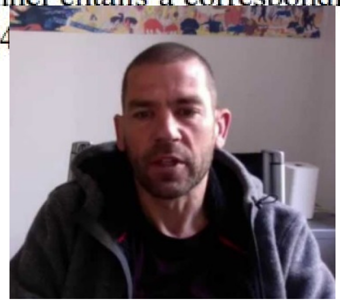
PHYSICAL REVIEW B VOLUME 55, NUMBER 2 1 JANUARY 1997-II

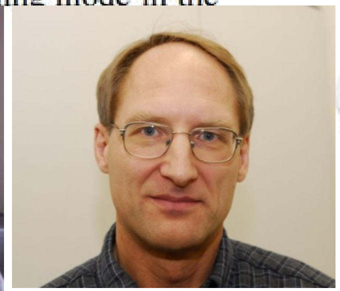
Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures

Particle-hole symmetry

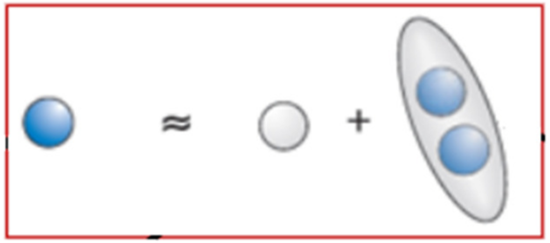
Alexander Altland and Martin R. Zirnbauer
Institut für Theoretische Physik, Universität zu Köln, Zülpicherstrasse 77, 50937 Köln, Germany
(Received 4 March 1996)

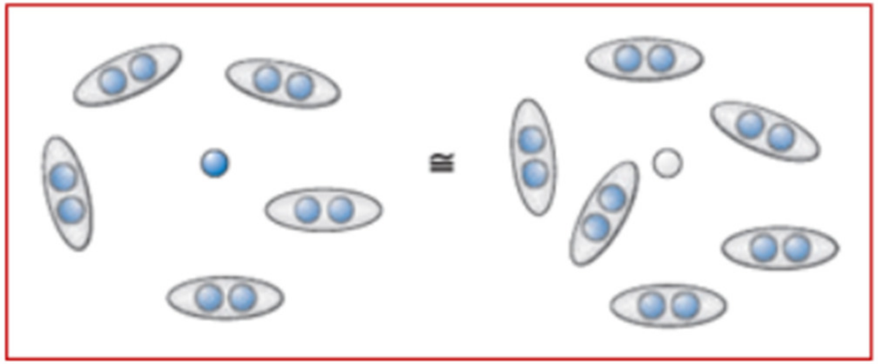
Normal-conducting mesoscopic systems in contact with a superconductor are classified by the symmetry operations of time reversal and rotation of the electron's spin. Four symmetry classes are identified, which correspond to Cartan's symmetric spaces of type *C*, *C*I, *D*, and *D*III. A detailed study is made of the systems where the phase shift due to Andreev reflection averages to zero along a typical semiclassical single-electron trajectory. Such systems are particularly interesting because they do not have a genuine excitation gap but support quasiparticle states close to the chemical potential. Disorder or dynamically generated chaos mixes the states and produces forms of universal level statistics different from Wigner-Dyson. For two of the four universality classes, the *n*-level correlation functions are calculated by the mapping on a free one-dimensional Fermi gas with a boundary. The remaining two classes are related to the Laguerre orthogonal and symplectic random-matrix ensembles. For a quantum dot with a normal-metal–superconducting geometry, the weak-localization correction to the conductance is calculated as a function of sticking probability and two perturbations breaking time-reversal symmetry and spin-rotation invariance. The universal conductance fluctuations are computed from a maximum-entropy *S*-matrix ensemble. They are larger by a factor of 2 than what is naively expected from the analogy with normal-conducting systems. This enhancement is explained by the doubling of the number of slow modes: owing to the coupling of particles and holes by the proximity to the superconductor, every cooperon and diffusion mode in the advanced-retarded channel entails a corresponding mode in the advanced-advanced (or retarded-retarded) channel. [S0163-1829(97)04



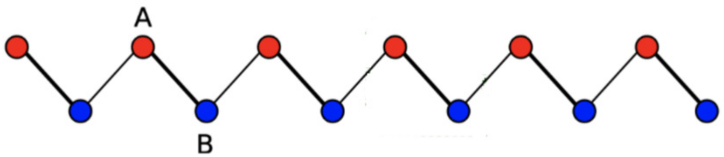


- Particle-hole symmetry of superconductor



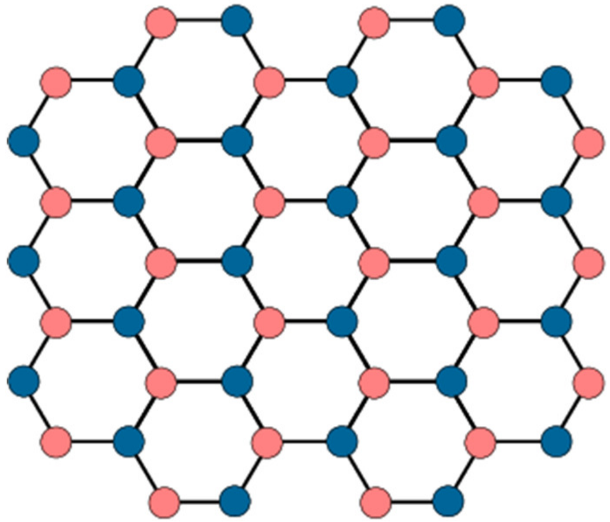


- Chiral symmetry (or sublattice symmetry)



NN coupling only

$$H_k = \begin{pmatrix} 0 & h_k \\ h_k^\dagger & 0 \end{pmatrix}$$



Symmetries of a Hamiltonian

- Unitary symmetry (translation, rotation, reflection …)
  Decompose H to irreducible blocks

- Beyond unitary symmetry
  (1) time-reversal symmetry (anti-unitary)

$$THT^{-1} = H, \quad T = U_T K$$
$$\rightarrow U_T H_k^* U_T^{-1} = H_{-k}$$

TRS = { 0 no TRS; +1 TRS with $T^2 = 1$ (integer spin); -1 TRS with $T^2 = -1$ (half-odd integer spin) }

  (2) particle-hole symmetry (anti-unitary)

$$PHP^{-1} = -H, \quad P = U_P K$$
$$\rightarrow U_P H_k^* U_P^{-1} = -H_{-k}$$

PHS = { 0 no PHS; +1 PHS with $P^2 = 1$ (odd parity: p-wave); -1 PHS with $P^2 = -1$ (even parity: s-wave) }

  (3) TRS x PHS = chiral symmetry (unitary) S=TP

$$TPH_k (TP)^{-1} = -H_k$$

S^2=1 ,-1 (chose +1)

Unitary, but not the usual one

- Any unitary operator that anticommutes with the band Hamiltonian $SH(k)S^{-1} = -H(k)$ qualifies as a chiral symmetry.

Classifying topological insulator/superconductor
using AZ classes

PHYSICAL REVIEW B **78**, 195125 (2008)

Classification of topological insulators and superconductors in three spatial dimensions

笠真生　　古崎 昭
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[2]*Condensed Matter Theory Laboratory, RIKEN, Wako, Saitama 351-0198, Japan*
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(Received 11 April 2008; revised manuscript received 13 September 2008; published 26 November 2008)

- Wigner-Dyson (1951 -1963) : "three-fold way" complex nuclei
- Verbaarschot (1992 -1993) chiral phase transition in QCD
- Altland-Zirnbauer (1997) : "ten-fold way" mesoscopic SC systems

3 internal symmetries

	Cartan's label	TRS	PHS	SLS	$d=1$	$d=2$	$d=3$	
Standard	A (unitary)	0	0	0	-	$\mathbb{Z}$	-	IQH,AQH
(Wigner-Dyson)	AI (orthogonal)	+1	0	0	-	-	-	
	AII (symplectic)	−1	0	0	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$	2D/3D TI
Chiral	AIII (chiral unitary)	0	0	1	$\mathbb{Z}$	-	$\mathbb{Z}$	
(sublattice)	BDI (chiral orthogonal)	+1	+1	1	$\mathbb{Z}$	-	-	SSH (with 3 symm)
	CII (chiral symplectic)	−1	−1	1	$\mathbb{Z}$	-	$\mathbb{Z}_2$	
BdG	D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	-	Kitev chain Chiral p-wave
	C	0	−1	0	-	$\mathbb{Z}$	-	
	DIII	−1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	Helical p-wave He-3 B
	CI	+1	−1	1	-	-	$\mathbb{Z}$	

Bogoliubov de Gennes

- (3x3-1)+2=10
- 5 non-trivial classes in each dimension
- Combined with unitary symm? → TCI… etc

Periodic table for topological insulators and superconductors

Alexei Kitaev

California Institute of Technology, Pasadena, CA 91125, U.S.A.

AIP Conf Proc 2009

Abstract. Gapped phases of noninteracting fermions, with and without charge conservation and time-reversal symmetry, are classified using Bott periodicity. The symmetry and spatial dimension determines a general universality class, which corresponds to one of the 2 types of complex and 8 types of real Clifford algebras. The phases within a given class are further characterized by a topological invariant, an element of some Abelian group that can be 0, $\mathbb{Z}$, or $\mathbb{Z}_2$. The interface between two infinite phases with different topological numbers must carry some gapless mode. Topological properties of finite syste are described in terms of K-homology. This classification is robust with respect to disorder, provided electron states near Fermi energy are absent or localized. In some cases (e.g., integer quantum Hall systems) the K-theoretic classification is sta to interactions, but a counterexample is also given.

complex class period 2

Real class Period82

Symmetry				d							
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	−1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	−1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	−1	−1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	−1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	−1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

IQHE, AQHE

SSH', π-flux state

TCI

SSH

Kitaev chain, chiral p-wave (spinless)

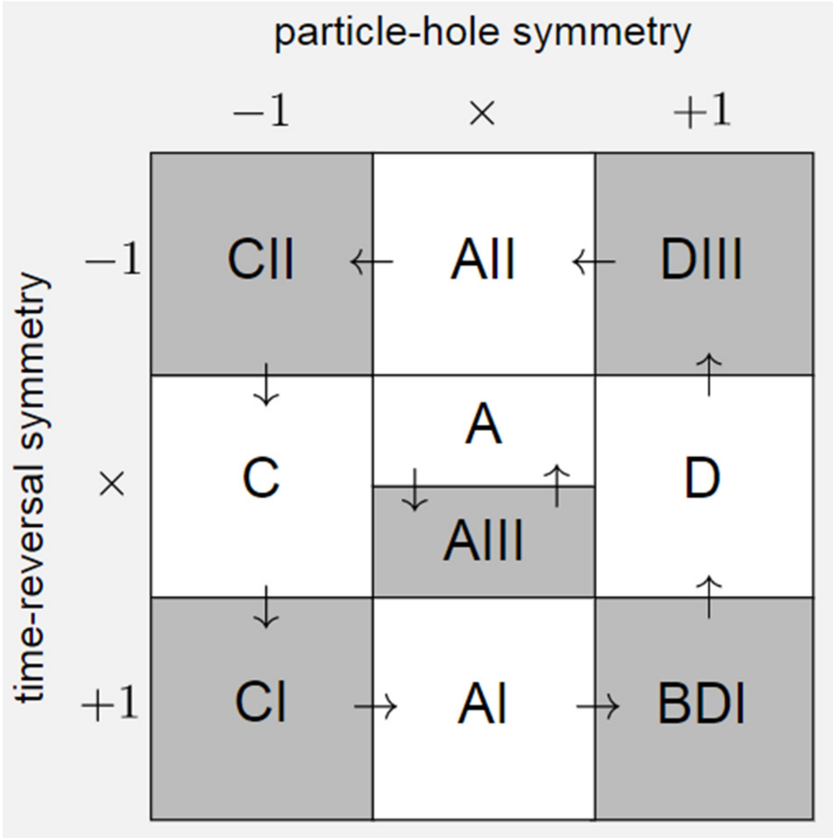
helical p-wave (spinful), He 3

2D/3D TI

d+id, d-id SC

d_{xy}, d_{x2-y2} singlet SC

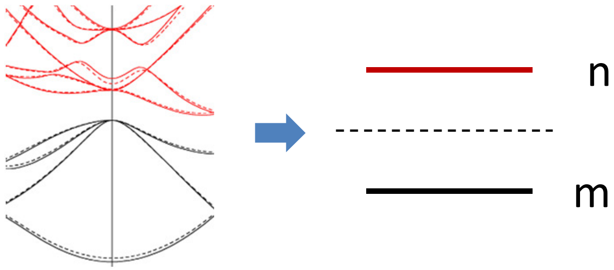
Bott periodicity



Grey: chiral

Topology of lattice system

"Flattened" Hamiltonian Q_k



Space of Q_k

Grassmannian:

$$G_{m+n,m}(C) = \frac{U(m+n)}{U(m)U(n)}$$

For chiral system,

$$Q_k = \begin{pmatrix} 0 & q_k \\ q_k^\dagger & 0 \end{pmatrix}$$

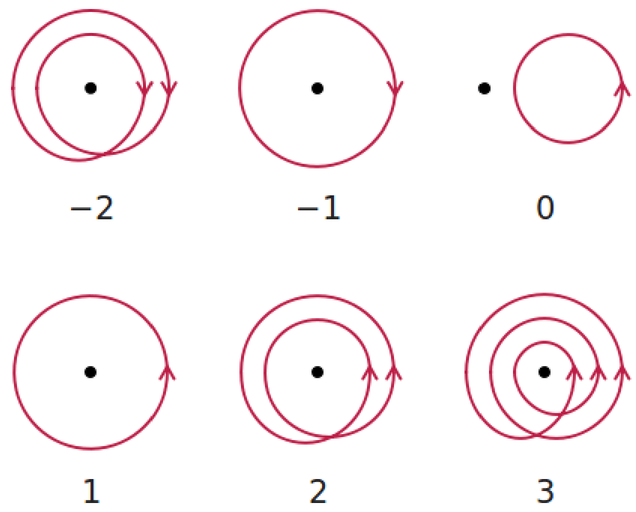
TABLE II Altland-Zirnbauer classes

	Cartan's label	T	P	S	1d	2d	3d	Space of Hamiltonians
Standard	A (unitary)	0	0	0	0	Z	0	$\{Q_k \in G_{m+n,m}(\mathbb{C})\}$
(Wigner-Dyson)	AI (orthogonal)	+1	0	0	0	0	0	$\{Q_k \in G_{m+n,m}(\mathbb{C}) \mid Q_k^* = Q_{-k}\}$
	AII (symplectic)	−1	0	0	0	Z_2	Z_2	$\{Q_k \in G_{2m+2n,2m}(\mathbb{C}) \mid i\sigma_y Q_k^*(-i\sigma_y) = Q_{-k}\}$
Chiral	AIII (chiral unitary)	0	0	1	Z	0	Z	$\{q_k \in U(m)\}$
(sublattice)	BDI (chiral orthogonal)	+1	+1	1	Z	0	0	$\{q_k \in U(m) \mid q_k^* = q_{-k}\}$
	CII (chiral symplectic)	−1	−1	1	Z	0	Z_2	$\{q_k \in U(2m) \mid i\sigma_y q_k^*(-i\sigma_y) = q_{-k}\}$
BdG	D	0	+1	0	Z_2	Z	0	$\{Q_k \in G_{2m,m}(\mathbb{C}) \mid \tau_x Q_k^* \tau_x = -Q_{-k}\}$
(superconductor)	C	0	−1	0	0	Z	0	$\{Q_k \in G_{2m,m}(\mathbb{C}) \mid \tau_y Q_k^* \tau_y = -Q_{-k}\}$
	DIII	−1	+1	1	Z_2	Z_2	Z	$\{q_k \in U(2m) \mid q_k^T = -q_{-k}\}$
	CI	+1	−1	1	0	0	Z	$\{q_k \in U(m) \mid q_k^T = q_{-k}\}$

Homotopy theory

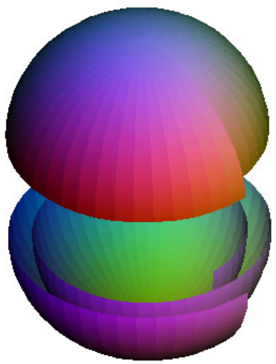
- Winding S^1 around S^1: winding number

$\pi_1(S^1)=Z$

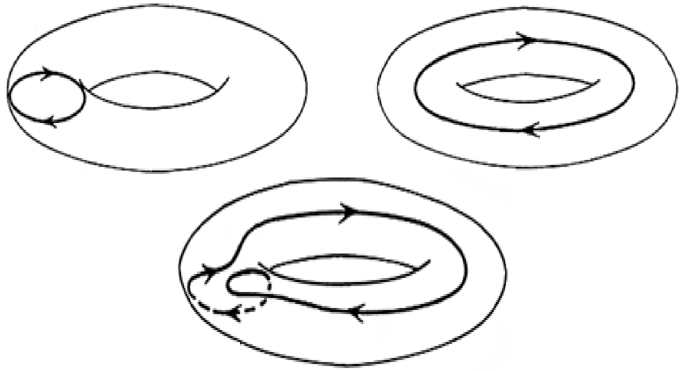


- Wrapping S^1 around S^2

$\pi_1(S^2)=0$



$\pi_2(S^2)=Z$



$\pi_1(T^2)=Z\times Z$

Topological numbers of complex classes

- Class A in *even* dim: Chern number

$$\pi_1(G_{m+n,m}(\mathbb{C})) = 0,$$
$$\pi_2(G_{m+n,m}(\mathbb{C})) = Z,$$
$$\pi_3(G_{m+n,m}(\mathbb{C})) = 0 \cdots$$

Note:

$$\pi_3(G_{2,1}(\mathbb{C})) = Z$$

$$C_n = \frac{1}{n!}\int_{T^d} \mathrm{tr}\left(\frac{i\mathsf{F}}{2\pi}\right)^n$$

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- Class AIII in *odd* dim (with chiral symm): winding number

$$\pi_{d\in\mathrm{odd}}((U(n)) = Z,$$
$$\pi_{d\in\mathrm{even}}((U(n)) = 0,$$

$$Q_k = \begin{pmatrix} 0 & q_k \\ q_k^\dagger & 0 \end{pmatrix}$$

$$\nu_{2n+1} = \frac{(-1)^n n!}{(2n+1)!}\int_{T^{2n+1}} \left(\frac{i}{2\pi}\right)^{n+1} \mathrm{tr}\left[(\mathsf{q}^{-1}d\mathsf{q})^{2n+1}\right]$$

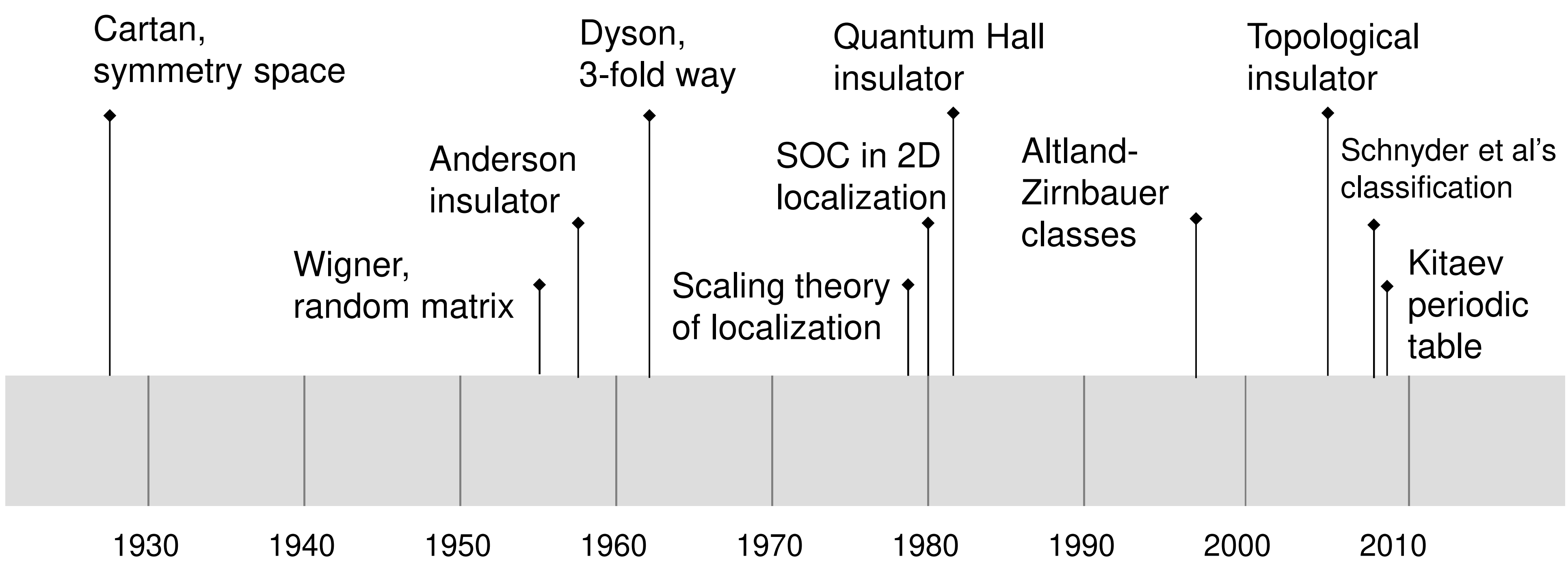
- The Z numbers in real classes are also related to these two complex class period 2

Real class Period82

AZ	Θ	Ξ	Π	d = 1	2	3	4	5	6	7	8	
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	IQHE, AQHE
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	SSH'
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	TCI
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	SSH
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	Kitaev chain, chiral p-wave (spinless)
DIII	−1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	helical p-wave (spinful), He 3
AII	−1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	2D/3D TI
CII	−1	−1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	
C	0	−1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	d+id, d-id SC
CI	1	−1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	d_{xy}, d_{x2-y2} singlet SC

Dimensional reduction (Lect 11)

A brief time-line of periodic table (for non-interacting fermions)



The garden of topological phases

- **Symmetry-protected topological (SPT) phase**

	Fermion	**Boson**
Non-interacting	• Integer quantum Hall effect • Topological insulator • … 	
Interacting	• … •	• Bosonic TI • Bosonic SC • …

spin

- Haldane's odd integer-spin chain
- …

- **Topological-order phase**

Strongly Interacting

- Fractional quantum Hall effect
- Chiral spin liquid
- Z_2 spin liquid (toric code)
- …

(***degenerate GND state, fractional QP, long-range entanglement***)